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Spontaneous Magnetization of the Ising Model on a 4–8 Lattice

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The spontaneous magnetization of the Ising model on a 4–8 lattice with six different coupling constants and two different magnetic moments is studied. A formula for the spontaneous magnetization is proposed. The result agrees with the exact low-temperature series expansions up to the 12th order.

KEY WORDS: Ising model; spontaneous magnetization; 4–8 lattice; series expansions.

1. INTRODUCTION

The spontaneous magnetization of the Ising model on a rectangular lattice was first obtained by Onsager in 1949.[1] He announced his famous result as a conference remark but never published his derivation. Yang[2] was the first to publish a derivation of the spontaneous magnetization on a square lattice. Chang[3] immediately generalized Yang's result to a rectangular lattice. In 1960, Syozi and Naya[4] made a conjecture for the spontaneous magnetization on a generalized square lattice whose unit cell has four different coupling constants.

The calculation of Yang was very complicated. A simple derivation was discovered by Montroll *et al.*[5] in 1963. The spontaneous magnetization was obtained as the limiting value of an infinite Toeplitz determinant. Their method was used by Lin and Fang[6] in an attempt to derive the spontaneous magnetization on a 4–8 lattice, which includes the generalized square lattice as a special case. The conjecture of Syozi and Naya was confirmed. Shinmi and Huckably[7] studied a three-component system on the square lattice. Their model is equivalent to the Ising model on a 4–8 lattice. Recently a mistake in the derivation of Lin and Fang was

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discovered and their result does not agree with the exact low-temperature series expansions.[18] It turns out that their derivation is valid only in the special case of a generalized square lattice, and the method of Montroll *et al.* for a square lattice should be modified for a 4–8 lattice such that the matrix elements of the Toeplitz determinant should be replaced by matrices.[9] Unfortunately it is not possible to calculate such block Toeplitz determinant in general.[10]

In the usual derivation of the spontaneous magnetization of the Ising model,[1-6] the magnetic moments are assumed to be the same for all spins. In the present paper, we study the spontaneous magnetization on a 4–8 lattice with six different coupling constants and two different magnetic moments. A formula for the spontaneous magnetization is proposed. Our result agrees with series expansions up to the 12th order and reduces to the exact solution in several special cases. Our conjecture generalizes the one made by Lin *et al.*[8] who studied a 4–8 lattice with four different coupling constants and one magnetic moment.

2. SERIES EXPANSIONS

Consider the Ising model of N spins on a 4–8 lattice with six coupling constants (J_1, J_2, J'_1, J'_2, J'_3, J'_4) and two magnetic moments (m, m') as shown in Fig. 1. Each spin located on the horizontal bonds carries a

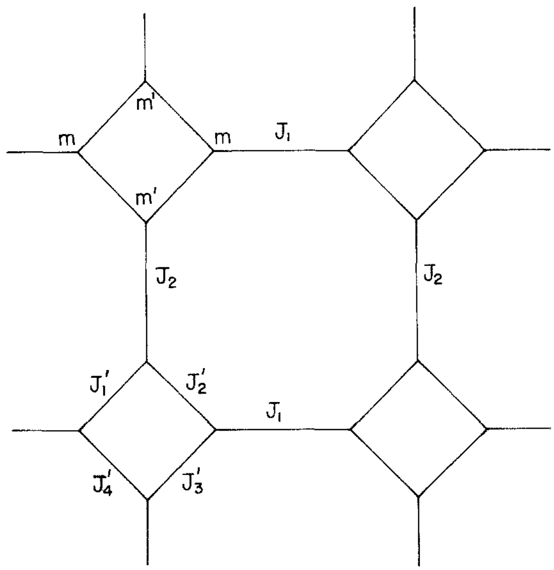

Fig. 1. A 4–8 lattice with six different coupling constants and two different magnetic moments.

magnetic moment m and each spin on the vertical bonds carries a magnetic moment m'.

We have calculated the exact low-temperature series expansions for the spontaneous magnetization M up to the 12th order. The method of series expansions is well known.[11] When each spin carries one unit of magnetic moment, the spontaneous magnetization is

$$M = 1 - 2 \sum_{r=1}^{\infty} r g_r \tag{1}$$

where g_r is the coefficient of N in the expansion of $F_r(N)$ in powers of N, and F_r is related to the partition function Z_N, the ground state energy E_0 and the magnetic field H by

$$Z_N \exp(E_0/kT) = 1 + \sum_{r=1}^{\infty} \mu^r F_r(N), \qquad \mu = \exp(-2H/kT) \tag{2}$$

The formula (1) can be generalized to the case where the spins may carry different magnetic moments. A spin carrying m units of magnetic moment may be considered as m spins, each carrying one unit of magnetic moment, coupled together by infinitely strong coupling constants. The interactions are assumed to be ferromagnetic ($J > 0$). The special case of $J'_1 = J'_3$, $J'_2 = J'_4$, $m = m'$ was discussed by Lin *et al.*[8]

We define

$$\begin{aligned}
x_i &= \exp(-2J'_i/kT) & & \\
y &= \exp(-2J_1/kT), & y' &= \exp(-2J_2/kT) \\
a &= 1 + x_1 x_2 x_3 x_4, & b &= x_1 x_2 + x_3 x_4 \\
c &= x_1 x_4 + x_2 x_3, & d &= x_1 x_3 + x_2 x_4 \\
g &= x_1 x_2 x_3 x_4, & h &= x_1^2 + x_2^2 + x_3^2 + x_4^2
\end{aligned} \tag{3}$$

The series expansions are (M is normalized to 1 at zero temperature)

$$M = 1 + \sum_{i=3}^{\infty} M_i \tag{4}$$

where

$$M_3 = -(myc + m'y'b)/(m + m')$$

$$M_4 = -2g - 2(yy')^2 - 2yy'd$$

$$M_5 = -2(d + 2yy')(yb + y'c) + d(myb + m'y'c)/(m + m')$$

$$\begin{aligned}
M_6 = & -4\sum (x_i x_j x_k)^2 - 4(yy')^2(y^2+y'^2) - 6yy'bc \\
& - 6yy'(y^2+y'^2)d - 2y^2(b^2+d^2) - 2y'^2(c^2+d^2) \\
& + [m(yc)^2 + m'(y'b)^2]/(m+m') + 2g(my^2+m'y'^2)/(m+m')
\end{aligned}$$

$$\begin{aligned}
M_7 = & -6(by^3+cy'^3)d - 2[bc+4yy'(y^2+y'^2)](yb+y'c) \\
& - 2[d^2+2(yy')^2+6dyy'](yc+y'b) \\
& + [7g-d^2+2dyy'+2(yy')^2](myc+m'y'b)/(m+m') \\
& - bc(myb+m'y'c)/(m+m')
\end{aligned}$$

$$\begin{aligned}
M_8 = & 2g^2 - 6g\sum (x_i x_j)^2 - 22(yy')^4 - 6(yy')^2(y^4+y'^4) \\
& - 8(y^2+y'^2)\,bcd - 4y^4(b^2+d^2) - 4y'^4(c^2+d^2) \\
& - 2gh(my^2+m'y'^2)/(m+m') - 10dyy'(y^4+y'^4) \\
& - 46d(yy')^3 + 2dyy'(3g-3d^2-5b^2-5c^2) \\
& + 4yy'(myc+m'y'b)(yb+y'c)/(m+m') - 30(yy'd)^2 \\
& - 18bcyy'(y^2+y'^2) - 16(b^2+c^2)(yy')^2 \\
& + 2dyy'(mb^2+m'c^2)/(m+m')
\end{aligned}$$

$$\begin{aligned}
M_9 = & 2(myc+m'y'b)(b^2y^2+c^2y'^2-3hg)/(m+m') \\
& - 6g(mcy^3+m'by'^3)/(m+m') - (mc^3y^3+m'b^3y'^3)/(m+m') \\
& + d(myb+m'y'c)[b^2+c^2+d^2-8g-2yy'(d+yy')]/(m+m') \\
& - 2(by'+cy)(d+yy')[bc+(3d+4yy')(y^2+y'^2)] \\
& + (myc+m'y'b)[3bc+2(y^2+y'^2)(d+yy')](d+2yy')/(m+m') \\
& - 2(by+cy')[2d(b^2+c^2+d^2-2g) \\
& + (5d+6yy')(y^4+y'^4) + 3bc(y^2+y'^2) \\
& + 21d^2yy' + 53dy^2y'^2 + 36y^3y'^3 + 6(b^2+c^2-g)yy']
\end{aligned}$$

$$\begin{aligned}
M_{10} = & (d^2+6g)(my^2b^2+m'y'^2c^2)/(m+m') \\
& - 2d(d+2yy')(by+cy')(myb+m'y'c)/(m+m') \\
& + (dyy'-10g+4d^2)(my^2c^2+m'y'^2b^2)/(m+m') \\
& + (my^2+m'y'^2)(2b^2c^2+6gd^2-40g^2-4gy^2y'^2-4gdyy')/(m+m') \\
& + 2(myc+m'y'b)(by+cy')[bc+(y^2+y'^2)(3d+4yy')]/(m+m') \\
& - 6d(d+yy')(y^6+y'^6) - 44g^2h - 70bcd^2yy' - 14bcd(y^4+y'^4)
\end{aligned}$$

$$
\begin{aligned}
&-6(y^4+y'^4)(b^2y^2+c^2y'^2+5bcyy')+28g[bcd+hyy'(d+yy')]\\
&+[2g-4(b^2+c^2+d^2)](bcd-gh)\\
&+4(3g-b^2-c^2)(b^2y^2+c^2y'^2+3bcyy')\\
&-(b^2y^2+c^2y'^2)(18d^2+84dyy'+92y^2y'^2)-254bcdy^2y'^2-248bcy^3y'^3\\
&-2(y^2+y'^2)[2d^2(b^2+c^2+d^2)\\
&+8(b^2+c^2)\,yy'(d+yy')-6gd^2+28y^4y'^4\\
&+dyy'(28g+18d^2)+y^2y'^2(36g+67d^2+79dyy')]
\end{aligned}
$$

and we sum over i, j, k (which are different from each other) from 1 to 4. The higher order terms are very complicated. In the special case of $m = m'$, the spontaneous magnetization is given incorrectly by Lin and Fang as[6]

$$M_0 = (1-k^2)^{1/8} \tag{5}$$

where

$$
\begin{aligned}
k &= 4p/q(1-y^2)(1-y'^2)\\
p^2 &= [(1-y^2)(1-y'^2)]^2 x_1x_2x_3x_4(1-x_1^2)(1-x_2^2)(1-x_3^2)(1-x_4^2)\\
&\quad+(a+by'+cy+dyy')(ay+byy'+c+dy')(ay'+b+cyy'+dy)\\
&\quad\times(ayy'+by+cy'+d)\\
q^2 &= (a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)\\
&\quad+16x_1x_2x_3x_4(1-x_1^2)(1-x_2^2)(1-x_3^2)(1-x_4^2)
\end{aligned}
$$

We rewrite M in the form $M = M_0F$. The series expansions of F up to the 12th order are

$$F = 1 + [mf(y, x_1, x_2, x_3, x_4) + m'f(y', x_1, x_4, x_3, x_2)]/(m+m') \tag{6}$$

where

$$f(y, x_1, x_2, x_3, x_4) = \sum_{i=3}^{\infty} f_i$$

$$
\begin{aligned}
f_3 &= -yc\\
f_4 &= 0\\
f_5 &= ybd\\
f_6 &= y^2(c^2+2g)
\end{aligned}
$$

$$f_7 = yc(5g - b^2 - d^2)$$

$$f_8 = -2y^2(bcd + gh)$$

$$f_9 = -cy^3(6g + c^2) + y[bd(b^2 + 2c^2 + d^2 - 6g) - 4gch]$$

$$f_{10} = y^2[b^2d^2 + (b^2 + d^2)(2c^2 + 6g) - 8gc^2 - 36g^2]$$

$$f_{11} = 3y^3bd(c^2 + 2g) + 6y^3cgh - y[c(b^4 + d^4) + c(b^2 + d^2)(c^2 - 19g) + 5b^2cd^2 - 4bdgh - 8c^3g + 77cg^2]$$

$$f_{12} = y^4(c^4 + 12gc^2 + 6g^2) + y^2[34bcdg - 2g^2h - 6bcd(b^2 + c^2 + d^2) + 12c^2gh]$$

We propose an exact expression for F:

$$(m + m')F = m[1 + 4y(ac - bd)\Delta + y^2(1 - 2q\Delta)]^{-1/2} + m'[1 + 4y'(ab - cd)\,\Delta' + y'^2(1 - 2q\Delta')]^{-1/2} \tag{7}$$

where

$$\Delta = (a^2 + c^2 - b^2 - d^2 + q)^{-1}, \qquad \Delta' = (a^2 + b^2 - c^2 - d^2 + q)^{-1}$$

In the special case of $x_1 = x_3 = x$ and $x_2 = x_4 = x'$, we have[12]

$$(m + m')F = m[1 + 4yxx'/(1 + x^2)(1 + x'^2)]^{-1/2} + m'[1 + 4y'xx'/(1 + x^2)(1 + x'^2)]^{-1/2} \tag{8}$$

The expression (7) agrees with the series expansions (6) up to the 12th order and we have $F = 1$ in the limit of $y = y' = 0$ (generalized square lattice). M vanishes at the critical temperature determined by $k = 1$.[13]

The spontaneous magnetization can be expressed in terms of

$$\begin{aligned} X_i &= \tanh(J_i'/kT) \\ Y_i &= \tanh(J_i/kT) \\ A &= 1 + X_1X_2X_3X_4 \\ B &= X_1X_2 + X_3X_4 \\ C &= X_1X_4 + X_2X_3 \\ D &= X_1X_3 + X_2X_4 \end{aligned} \tag{9}$$

such that

$$M = (1 - k^2)^{1/8}F \tag{10}$$

where

$$k = P/4Y_1Y_2Q$$

with

$$\begin{aligned}P^2 = {} & 16(Y_1Y_2)^2X_1X_2X_3X_4(X_1^2-1)(X_2^2-1)(X_3^2-1)(X_4^2-1)\\ & + (A+BY_1+CY_2+DY_1Y_2)(A+BY_1-CY_2-DY_1Y_2)\\ & \times (A-BY_1+CY_2-DY_1Y_2)(A-BY_1-CY_2+DY_1Y_2)\\ Q^2 = {} & ABCD + X_1X_2X_3X_4(X_1^2-1)(X_2^2-1)(X_3^2-1)(X_4^2-1)\end{aligned}$$

and

$$\begin{aligned}(m+m')F = {} & m\left(1+2y\frac{AC-BD}{AC+BD+2Q}+y^2\frac{AC+BD-2Q}{AC+BD+2Q}\right)^{-1/2}\\ & + m'\left(1+2y'\frac{AB-CD}{AB+CD+2Q}+y'^2\frac{AB+CD-2Q}{AB+CD+2Q}\right)^{-1/2}\end{aligned}$$

3. EXACTLY SOLUBLE CASES

There are several cases where our conjecture is exactly correct.

Case 1. $J_1 = J_2 = \infty$ $(y = y' = 0)$. This case corresponds to a generalized square lattice with two different magnetic moments. Syozi and Naya[4] considered the special case of $m = m'$ and their conjecture was confirmed by Lin and Fang.[6] We shall prove that their result is in fact valid for arbitrary m and m' (i.e., $F = 1$).

A special case of $m' = 0$ and $J_4 = 0$ was considered by Naya,[14] who called such lattice a semiferromagnetic honeycomb lattice. He showed that the normalized spontaneous magnetization of the semiferromagnetic lattice is exactly the same as that of the normal ferromagnetic lattice.

The generalized square lattice consists of two sublattices associated with different magnetic moments. We calculate the normalized spontaneous magnetization by the method of series expansions. Each graph drawn on the lattice corresponds to a term in the series

$$f(1,2,3,4)(am+bm')/(m+m') \tag{11}$$

where

$$f(1,2,3,4) = f(x_1, x_2, x_3, x_4)$$

and a (b) is the number of sites associated with magnetic moment m (m') in that graph. The integers a and b remain the same if we reflect the graph along either a vertical or a horizontal line. Such graphs taken together correspond to

$$[f(1,2,3,4)+f(2,1,4,3)+f(4,3,2,1)+f(3,4,1,2)](am+bm')(m+m')^{-1} \tag{12}$$

The generalized square lattice is invariant under exchange of (m, m'), $(1, 3)$, $(2, 4)$, as shown in Fig. 2. Therefore the expression (12) in the series expansion is always associated with the exchanged expression such that the sum is proportional to

$$[(am+bm')+(am'+bm)]/(m+m')=a+b \tag{13}$$

which is independent of m and m'.

Case 2. $J_4'=\infty$ $(x_4'=0)$. When $J_4'=\infty$ the 4–8 lattice reduces to a special case of the 3–12 lattice, as shown in Fig. 3. Recently an exact solution for the spontaneous magnetization of the Ising model on a general 3–12 lattice with nine coupling constants and six magnetic moments was derived by Lin and Chen,[15] who generalized the result of Huckaby[16] on an isotropic 3–12 lattice with two coupling constants and one magnetic moment. Our conjecture is correct in this case and we have

$$(m+m')F=m(1+y\Delta)^{-1}+m'(1+y'\Delta')^{-1} \tag{14}$$

where

$$\Delta(A,B,C,D)=[(AC)^{1/2}-(BD)^{1/2}]/[(AC)^{1/2}+(BD)^{1/2}]$$
$$\Delta'=\Delta(A,C,B,D)$$

Case 3. $J_4'=0$ $(x_4'=1)$. When $J_4'=0$, the 4–8 lattice reduces to a decorated honeycomb lattice, as shown in Fig. 4. The decorated

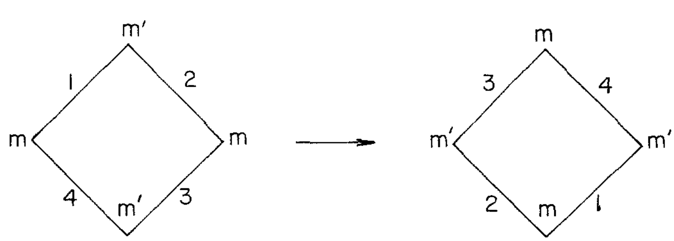

Fig. 2. The generalized square lattice is invariant under the exchange of (m, m'), $(1, 3)$, $(2, 4)$.

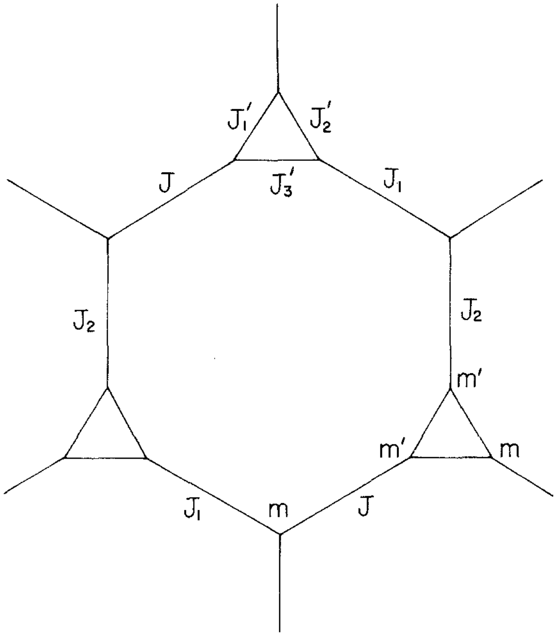

Fig. 3. When $J'_4 = \infty$, the 4–8 lattice is equivalent to a special case of the 3–12 lattice with $J = \infty$.

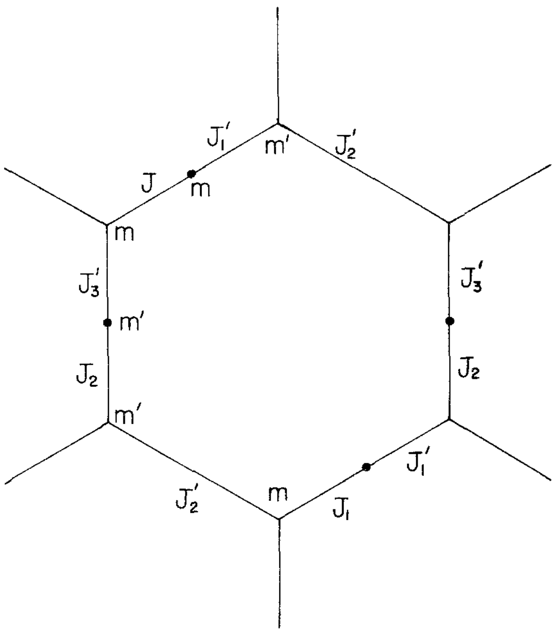

Fig. 4. When $J'_4 = 0$, the 4–8 lattice is reduced to a decorated honeycomb lattice.

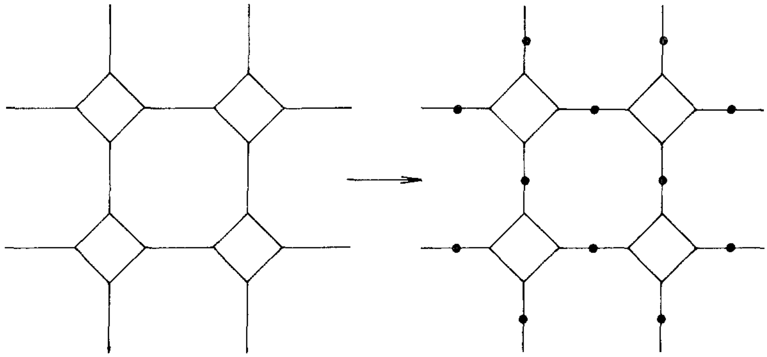

Fig. 5. The 4–8 lattice is transformed into a decorated 4–8 lattice such that only the decorating spins in the new lattice carry magnetic moments.

honeycomb lattice is related to the normal honeycomb lattice by a decoration transformation.[17] Our conjecture agrees with the exact solution and we have

$$(m+m')F=m(1+yx_1)^{-1}+m'(1+y'x_3)^{-1} \tag{15}$$

Case 4. Here

$$(x_1-x_3)(x_2-x_4)=0$$

$$y=y'=-(x_1+x_3)(x_2+x_4)/(1+x_1x_3)(1+x_2x_4)$$

It is pointed out by Wu[18] that the 4–8 lattice reduces to a square lattice under certain conditions. To see this, let us first set $J_1=J_2$ and transform the 4–8 lattice into a decorated 4–8 lattice as shown in Fig. 5, such that only the decorating spins carry magnetic moments. Next we transform the decorated 4–8 lattice into a square lattice as shown in Fig. 6.

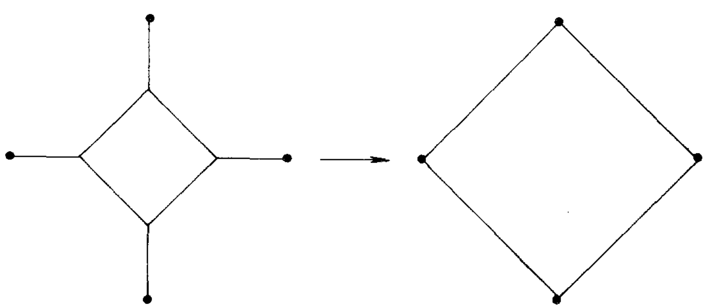

Fig. 6. A decorated 4–8 lattice can be transformed into a square lattice under certain conditions.

It can be shown[18] that the second transformation is possible, provided

$$\begin{gathered}(x_1 - x_3)(x_2 - x_4) = 0 \\ y = y' = -(x_1 + x_3)(x_2 + x_4)/(1 + x_1 x_3)(1 + x_2 x_4)\end{gathered} \tag{16}$$

In this case our conjecture is correct and we have

$$F = (1 - y^2)^{-1/2} \tag{17}$$

4. SUMMARY

We have proposed a formula for the spontaneous magnetization of the Ising model on a general 4–8 lattice with six different coupling constants and two different magnetic moments. The previous result by Lin and Fang is modified. Our conjecture is supported by the following evidence: (1) $M = 0$ at the exact critical temperature. (2) Our formula agrees with the exact low-temperature series expansions up to the 12th order. (3) Our result is exact in several special cases.

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